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Resistance and reflection phase distributions in short one-dimensional conductors

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Abstract. Resistance and reflection phase distributions are obtained from the exact asymptotic solution of the invariant embedding equations for arbitrary disorder, for samples of length $L \ll (2k_0)^{-1} < L_c$ (with k_0 the incident electron wavenumber and L_c the localization length). The resistance moments differ quantitatively from earlier results for the range $(2k_0)^{-1} \ll L \ll L_c$ and from results obtained by assuming uniformly distributed phases. The phase distribution reduces, for weak disorder, to a binary form with possible phases $\pi/2$ and $3\pi/2$. The conductance moments diverge more strongly than the results based on uniformly random phases in 1D.

The Landauer [1] transport formula defines the dimensionless resistance of a random one-dimensional (1D) conductor of length L in terms of the reflection amplitude $R(L)$ and transmission amplitude $T(L)$, for an incident plane wave:

$$\rho = \frac{|R(L)|^2}{|T(L)|^2} \qquad |R(L)|^2 + |T(L)|^2 = 1. \tag{1}$$

The invariant embedding method [2,3] addresses directly these emerging quantities and describes $R(L)$ and $T(L)$ by a closed set of non-linear differential equations. In particular the complex reflection amplitude $R \equiv R(L) = |R(L)| \exp(\mathrm{i}\theta(L))$ is given by (in units with $\hbar = m = 1$)

$$\mathrm{i}k_0 \frac{\mathrm{d}R}{\mathrm{d}L} = V(L)(1+R)^2 - 2k_0^2 R \tag{2}$$

where $V(L)$ is the random potential which we assume to have the Gaussian δ-correlated form

$$\langle V(L)V(L')\rangle = V_0^2\,\delta(L-L') \qquad \langle V(L)\rangle = 0 \tag{3}$$

and k_0 is the wavenumber of a plane wave incident at the edge $x = L$ of the conductor. The invariant embedding approach has provided detailed confirmation [3–5] of the non-self-averaging nature of the resistance [6] both for long lengths $L \gg L_c$ (with $L_c = k_0^2/V_0^2$ the localization length) and for short lengths $L \ll L_c$. For short sample lengths the mean resistance obeys Ohm's law $\langle\rho\rangle = L/L_c$, which implies diffusive motion [1] (quasi-metallic regime). Similar results have also been obtained with other quite different methods [7,8].

A puzzling feature of the treatments using the Landauer formula (1) [3–5,8] is that they all lead to divergent moments for the conductance $g = 1/\rho$ on a short length scale L which, in fact, supports the original suggestion of Landauer [1]. The divergent conductance

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moments have been attributed to Azbel [9] transmission resonances associated with zero-resistance realizations of the random potential. Surprising as they are, these results are not in conflict, however, with universal conductance fluctuations in metallic systems [10] which have been demonstrated for multi-channel systems such as two- and three-dimensional metals and so-called quasi-one-dimensional many-channel systems. We recall that the multichannel treatments use a simple generalization of the one-channel formula (1) [10, 11] which leads to finite conductances but which is invalid in the high-transmission limit in 1D.

The studies of resistance (conductance) fluctuations by means of the invariant embedding approach [3–5] (or by other methods [6–8]) usually rely on the so-called random-phase model (RPM); one assumes that the phase of the reflection amplitude ($\theta \equiv \theta(L)$ defined modulo 2π) is an independent random variable which is uniformly distributed between 0 and 2π. In order to put the above results for 1D systems on a firmer basis, it is, of course, desirable to check the validity of the RPM as thoroughly as possible. This seems particularly important for samples of length shorter than the localization length ($L \ll L_c$) for which the conductance moments diverge, as recalled above. Some time ago we studied the exact invariant embedding distribution of the phase in the low-reflection (quasi-metallic) regime ($L \ll L_c$) for $2k_0L \gtrsim 1$ [12]. We also calculated the low-order resistance moments using an exact statistical treatment of the actual solution (i.e. the solution obtained without further assumptions about phases) of the stochastic equation (2) for low reflection. In particular, while for $2k_0L \gtrsim 1$ the phase distribution is quite structured [12], for larger values ($(2k_0)^{-1} \ll L \ll L_c$) it becomes increasingly uniform and the resistance moments agree to within terms of order $1/(2k_0L)^2$ with the RPM moments.

Obviously, the analysis of the properties of the phase and of the resistance in the domain $L \ll (2k_0)^{-1}$ is also of interest, particularly in cases where L_c is close to its lower limit $(2k_0)^{-1}$ (which is known as the Ioffe–Regel limit). The purpose of this paper is precisely to discuss analytical results for the resistance moments and for the phase distribution obtained from equation (2) for $L \ll (2k_0)^{-1}$, for arbitrary disorder. The resistance moments differ only quantitatively (by a factor of $(2n-1)!!/n!$ for the nth moment) from the earlier results [12] for $2k_0L \gg 1$, while the phase distribution is qualitatively different (the phase distribution derived in [12] is clearly invalid for $2k_0L \ll 1$).

Part of the motivation for presenting these results is provided by a recent paper of Pradhan and Kumar [13] who argue unjustifiably, I believe, that statistical correlations between $\theta(L)$ and the random potential in equation (2) largely suppress conductance fluctuations, making the mean conductance finite in 1D.

It is convenient to eliminate the kinetic energy term on the right-hand side of equation (2) by defining $Q(L) = R(L)\exp(-2\mathrm{i}k_0L)$ which satisfies

$$\mathrm{i}k_0\frac{\mathrm{d}Q}{\mathrm{d}L} = V(L)\exp(-2\mathrm{i}k_0L)(1+\exp(2\mathrm{i}k_0L)Q)^2. \tag{4}$$

As mentioned above, we are interested in a domain of sample lengths

$$L \ll (2k_0)^{-1} \lesssim L_c \tag{5}$$

where the asymptotically exact solution of (4) (with $Q(L=0)=0$) is

$$Q(L) = -\frac{\mathrm{i}z(L)}{1+\mathrm{i}z(L)} \tag{6}$$

with

$$z(L) \equiv z = \frac{1}{k_0}\int_0^L \mathrm{d}L'\, V(L'). \tag{7}$$

This leads to very simple expressions for the resistance (1) and for the phase of the reflection amplitude in terms of the random variable z, namely

$$\rho = z^2 \tag{8}$$

$$\theta = \tan^{-1}\left(\frac{1}{z}\right) \tag{9}$$

which define the probability distributions of ρ and θ in terms of the distribution of z as

$$P_\rho(\rho; L) = \int_{-\infty}^{\infty} \mathrm{d}z\, P_z(z; L)\delta(\rho - z^2) \tag{10}$$

$$P_\theta(\theta; L) = \int_{-\infty}^{\infty} \mathrm{d}z\, P_z(z; L)\delta\left[\theta - \tan^{-1}\left(\frac{1}{z}\right)\right]. \tag{11}$$

The distribution of z is the inverse Fourier transform of the moment-generating function

$$\phi(k) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}k)^n}{n!}\langle z^n\rangle \tag{12}$$

where, from (7) and (3), $\langle z^{2n+1}\rangle = 0$, $n = 0, 1, 2, \ldots$, and

$$\langle z^{2n}\rangle = \langle \rho^n\rangle = 1 \times 3 \times 5 \times \cdots \times (2n-1)l^n \qquad n = 1, 2, 3, \ldots \tag{13}$$

where $l = L/L_c$. Equation (13) is readily obtained from the standard result for the average of a product of $2n$ Gaussian variables. From (12) and (13) it then follows that $P_z(z; L)$ is the Gaussian

$$P_z(z; L) = \frac{1}{\sqrt{2\pi l}} \exp\left(-\frac{z^2}{2l}\right). \tag{14}$$

By inserting (14) in (10) and (11) we obtain successively

$$P_\rho(\rho; L) = \frac{1}{\sqrt{2\pi l}} \frac{\exp(-\rho/2l)}{\sqrt{\rho}} \tag{15}$$

$$P_\theta(\theta; L) = \frac{1}{\sqrt{2\pi l}}\left(1 + \frac{1}{\tan^2\theta}\right)\exp\left(-\frac{1}{2l\tan^2\theta}\right). \tag{16}$$

We recall that the above results are valid for any strength of disorder for short sample lengths $L \ll (2k_0)^{-1} < L_c$.

The resistance moments (13) are qualitatively similar to the low-order moments ($n = 1, 2, 3$) found previously for weak disorder, for arbitrary values of $2k_0L$ [12]. In fact, the asymptotically exact moments (13) for $n = 1, 2, 3$ coincide with the leading terms of the expansions of our earlier results for $2k_0L \ll 1$. On the other hand, these moments differ only through the numerical coefficients from the moments

$$\langle \rho^n\rangle \simeq n!l^n \qquad n = 0, 1, 2, \ldots \tag{17}$$

which have been obtained previously, for $L \ll L_c$, from the invariant imbedding equations by arbitrarily assuming θ to be uniformly distributed [3, 4] (RPM). The short-scale moments (13) for any disorder differ qualitatively from the corresponding moments for strong disorder for long scales ($L \gg L_c$) whose exponential growth with L reflects the localization of electrons within domains of length L_c. Finally the resistance distribution (15) should be contrasted with the Poissonian distribution [14]

$$P_\rho(\rho; L) = \frac{1}{l}\exp\left(-\frac{\rho}{l}\right) \tag{18}$$

obtained from the RPM moments (17) for the quasi-metallic regime.

The phase distribution (16) defined in the domain $0 \leqslant \theta \leqslant 2\pi$ has peaks centred at $\theta = \pi/2$ and $\theta = 3\pi/2$ respectively. It is symmetric about $\theta = \pi$ where it goes to zero, as it does also at $\theta = 0$ and at $\theta = 2\pi$. The half-widths of the peaks are of order $l \ll 1$ which allows us to expand $1/\tan^2\theta$ into small deviations from the peak positions. This yields

$$P_\theta(\theta; L) \simeq \frac{1}{2}\frac{1}{\sqrt{2\pi l}}\left\{\exp\left[-\frac{1}{2l}\left(\theta - \frac{\pi}{2}\right)^2\right] + \exp\left[-\frac{1}{2l}\left(\theta - \frac{3\pi}{2}\right)^2\right]\right\} \tag{19}$$

whose limiting form for $l \to 0$ is

$$\lim_{l\to 0}[P_\theta(\theta; L)] = \frac{1}{2}\left[\delta\left(\theta - \frac{\pi}{2}\right) + \delta\left(\theta - \frac{3\pi}{2}\right)\right]. \tag{20}$$

Thus the actual phase distribution reduces asymptotically to a binary distribution with equiprobable possible values $\theta = \pi/2$ and $\theta = 3\pi/2$. Note that these values for the phase are also obtained if one linearizes (6) for $z \to 0$ (weak-disorder approximation) which leads to $Q = -\mathrm{i}z$ and to

$$\theta = -\mathrm{i}\ln\left(-\mathrm{i}\frac{z}{|z|}\right).$$

This yields $\theta = \pi/2$ for $\operatorname{sgn} z = z/|z| = -1$ and $\theta = 3\pi/2$ for $\operatorname{sgn} z = 1$. Since the values $\operatorname{sgn} z = \pm 1$ occur with equal Gaussian probabilities defined by (14), we recover equation (20).

It follows from the above discussion that for $L \ll (2k_0)^{-1} < L_c$ the phase distribution is strongly peaked around $\theta = \pi/2$ and $\theta = 3\pi/2$, due to the narrowness of the z distribution. On the other hand, in [12] we found that when $L_c \gg (2k_0)^{-1}$ the structure of the distribution *of θ evolves rapidly towards a uniform distribution as L is increased within the domain* $(2k_0)^{-1} \ll L \ll L_c$. This suggests that the length $L_\phi \sim (2k_0)^{-1}$ which separates the domain $L \ll (2k_0)^{-1}$ where $P_\theta(\theta; L)$ is highly structured from the domain $L \gg (2k_0)^{-1}$ where the structure of $P_\theta(\theta; L)$ is progressively washed out might be viewed as the analogue of the phase coherence length introduced by Stone *et al* [15] in the context of their study of the phase distribution in an Anderson tight-binding model.

Finally, in order to illustrate the strong interdependence (correlation) of the random resistance and phase variables, which is ignored in the RPM, we give their joint distribution. From (8) and (9) we have

$$P(\rho, \theta; L) = \left\langle \delta(\rho - z^2)\delta\left[\theta - \tan^{-1}\left(\frac{1}{z}\right)\right]\right\rangle$$

where the averaging over the z distribution (14) may be readily performed. Using (16) we get

$$P(\rho, \theta; L) = P_\theta(\theta; L)\delta\left(\rho - \frac{1}{\tan^2\theta}\right) \qquad 2k_0 L \ll 1.$$

In conclusion, we have presented an exact invariant embedding analysis of the statistical properties of the resistance and of the reflection phases for arbitrary disorder at length scales $L \ll (2k_0)^{-1} < L_c$ in the quasi-metallic regime in 1D. Significant quantitative differences between our results for the resistance moments (resistance distribution) and the corresponding results obtained by assuming the phases to be uniformly random have been noted. Our expression for the phase distribution reduces to a binary distribution with equiprobable values $\theta = \pi/2$ and $\theta = 3\pi/2$ for weak disorder.

From the distribution (15) it is clear that the exact conductance moments $\langle g^n \rangle = \langle \rho^{-n} \rangle$ of short ($L \ll (2k_0)^{-1}$) 1D conductors are infinite. This supports a similar conclusion reached in previous treatments [1, 3–5, 8] using the RPM. In fact, the logarithmic divergence of the mean conductance obtained from the RPM distribution (18) is now replaced by the stronger inverse square divergence which follows from (15). In particular, our exact treatment invalidates the conclusion of Pradhan and Kumar [13] concerning the finiteness of the mean conductance on short scales, based on a heuristic discussion of the embedding equations. These workers have also argued that the reflection phase for short conductors has two possible values ($\theta = \pi/2$ or $3\pi/2$), which has here been shown to follow exactly, for weak disorder, from a statistical treatment of the invariant embedding equation (2). The above treatment shows furthermore that these limiting phases are, in fact, the cause of the infinite values of the conductance moments since from (8) and (9) we have $g = \tan^2\theta$. This observation is consistent with the stronger divergence of the conductance moments when the phase is treated correctly compared with the results based on the resistance distribution (18) obtained by arbitrarily assuming uniformly distributed phases on all length scales.

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